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ON REACTIONS IN SYSTEMS WITH ONE-SIDED SUPPORTS

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Work done by systems with one-sided supports, point-type and continuous, is considered. In the case of discrete supports the problem is reduced to solving a system of differential equations with initial conditions, and in the case of an elastic support the problem reduces to solving linear equations with variable coefficients and the same boundary conditions as in the initial problem.

1. **Point supports.** A method is given for determining the reactions for the case of one-sided point supports.

If we denote by x_i the reaction in the i-th support and by y the distance between the body and the i-th support (which can be always assumed positive), then the problem reduces to solving a system of n differential equations

$$A\bar{x} + B\bar{y} + \bar{b} = 0 \tag{1.1}$$

with $2n$ unknown x_i and y_i, under the conditions that $x_i \geqslant 0$ when $y_i = 0$ and $y_i \geqslant 0$ when $x_i = 0$ (the solution must lie on at least one of the coordinate semi-axes). In the real technical problems the properties of the matrices A and B are such that a solution exists and is unique.

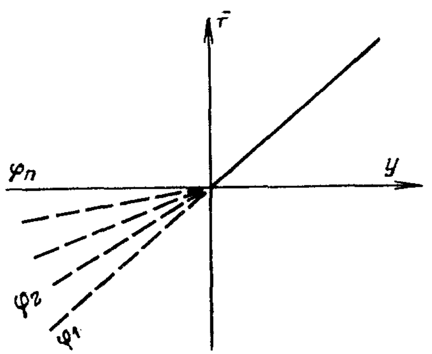

Fig. 1

First we perform a change of variables $x_i = \alpha u_i + \beta v_i, y_i = \gamma u_i + \delta v_i$, whereupon the system reduces to the form

$$C\bar{u} + D\bar{v} + \bar{h} = 0 \tag{1.2}$$

with the conditions that $\alpha u_i + \beta v_i \geqslant 0$ when $\gamma u_i + \delta v_i = 0$ and $\gamma u_i + \delta v_i \geqslant 0$ and $\alpha u_i + \beta v_i = 0$. From the geometrical point of view this means that the solution must lie on two rays, i. e. in a phase space, on a surface formed by two planes. Analytically it means that the solution is sought on the surface $\bar{u} = \bar{f}(\bar{v})$.

To obtain a solution in practice, we construct a system of surfaces depending on the parameter

$$\bar{u} = \bar{\varphi}(k, \bar{v}) \tag{1.3}$$

such that it becomes $\bar{u} = \bar{\varphi}(k_2, \bar{v}) \equiv \bar{f}(v)$, when $k = k_2$ and is transformed into the linear relation

$$\bar{u} = \bar{\varphi}(k_1, \bar{v}) \equiv a\bar{v} \tag{1.4}$$

when $k = k_1$. Function $\bar{u}$ must have continuous partial derivatives with respect to all its arguments, and this can be achieved by rounding slightly the angle shown in the Fig. 1.

Substituting (1.4) into (1.2), we can obtain $\bar{v}_1$ and $\bar{u}_1 = a\bar{v}_1$. Let us suppose that a solution of the system is known for some value of the parameter k. Let us determine this solution for $k + dk$. Using (1.2) we obtain

$$C\bar{\varphi}(k + dk,\ \bar{v} + d\bar{v}) + D(\bar{v} + d\bar{v}) + \bar{h} = C\bar{\varphi}(k, \bar{v}) +$$
$$C\left[\frac{\partial\varphi}{\partial k}(k, \bar{v})\, dk + \frac{\partial\bar{\varphi}(k, \bar{v})}{\partial\bar{v}}\, d\bar{v}\right] + D(\bar{v} + d\bar{v}) + \bar{h} + \ldots$$

where repeated dots denote the higher order terms equal to $C\bar{\varphi}_k(k, \bar{v})dk + [C\bar{\varphi}_{\bar{v}}'(k, \bar{v}) + D]d\bar{v} + 4Bn = 0$. Passing to the limit, we obtain the following system of nonlinear differential equations:

$$[C\bar{\varphi}_{\bar{v}}(k, \bar{v}) + D]\ \frac{d\bar{v}}{dk} + C\bar{\varphi}_k(k, \bar{v}) = 0$$

The value of $\bar{v}_1$ is known for $k = k_1$. We seek a solution $\bar{v}_2$ of this system for $k = k_2$ if, on integrating this system over the interval $[k_1, k_2]$ the matrix $E = C\bar{\varphi}_v(k, \bar{v}) + D$ never vanishes, then a solution exists and is unique when the usual constraints (such as Lipschitz conditions, etc.) are imposed on the system. The value of $\bar{u}_2$ can then be found from (1.3), namely $\bar{u}_2 = \bar{\varphi}(k_2, \bar{v}_2)$. Returning now to the variables $\bar{x}$ and $\bar{y}$, we obtain the solution of the system.

2. **Beam on an elastic support.** Equation of equilibrium of a beam resting on an elastic, one-sided support, can be written in the form $d^4y / dx^4 + f(y) = 0$ where y denotes the beam displacement and function $f(y)$ is depicted in Fig. 1. The problem here is that of satisfying numerous boundary conditions. To do this we include the function $f(y)$ in the family of functions $F(\varphi, y)$ which depends on the parameter φ, in such a manner that the value $\varphi = \varphi_1$ yields a linear relationship $F(\varphi_1, y) \equiv ky$ and $\varphi = \varphi_n$ a given function $f(y)$, i. e. $F(\varphi_n, y) \equiv f(ly)$.

The function $F(\varphi, y)$ must have continuous first order partial derivatives with respect to its arguments. It can be constructed by various methods when the corner in the neighborhood of the zero is rounded a little, which is of little practical significance.

If a solution of the problem $\partial^4 y(\varphi, x) / \partial x^4 + F(\varphi, x) = 0$ is known for some value of the parameter φ, then changing the latter by $d\varphi$ results in change in the value of the ordinate by dy. The resulting relation is

$$\frac{\partial^4 y(x, \varphi + d\varphi)}{\partial x^4} + F(\varphi + d\varphi, y + dy) = \frac{\partial^5 y(x, \varphi)}{\partial x^4 \partial\varphi}\, d\varphi +$$
$$\frac{\partial F(\varphi, y)}{\partial\varphi}\, d\varphi + \frac{\partial F(\varphi, y)}{\partial y}\, dy = 0$$

$$\left[\frac{\partial^5 y(\varphi, x)}{\partial x^4 \partial \varphi} + F_\varphi(\varphi, y)\right] d\varphi + F_y(\varphi, y)\, dy = 0$$

and the above partial differential equation can be solved by the method of straight lines.

Having specified a set of values for the parameter $\varphi = \varphi_1, \varphi_2, \ldots, \varphi_n,$ we can find the corresponding functions $y_1(x), y_2(x), \ldots, y_n(x)$ from the equations

$$\frac{d^4 y_1}{dx^4} - k y_1 = 0$$

$$\left[\frac{\partial^4 y_2}{\partial x^4} - \frac{\partial^4 y_1}{\partial x^4} + F_\varphi(\varphi_1, y_1)\right](\varphi_2 - \varphi_1) + F_y(\varphi_1, y_1)(y_2 - y_1) = 0$$

$$\left[\frac{\partial^4 y_n}{\partial x^4} - \frac{\partial^4 y_{n-1}}{\partial x^4} + F_\varphi(\varphi_{n-1}, y_{n-1})\right](\varphi_n - \varphi_{n-1}) + F_y(\varphi_{n-1}, y_{n-1})(y_n - y_{n-1}) = 0$$

The above equations are all linear and contain a variable coefficient in the right hand side. They must be solved using the same boundary conditions throughout. Since we deal here with small differences, they should be calculated directly, i. e. the system is best written in the following form:

$$\frac{d^4 y_1}{dx^4} + k y_1 = 0, \quad y_{k+1} = y_k + \Delta y_k$$

$$\frac{d^4(\Delta y_k)}{dx^4} + F_y(\varphi_k, y_k)\Delta y_k = -F_\varphi(\varphi_k, y_k)(\varphi_{k+1} - \varphi_k), \; k = 1, 2, \ldots, n-1$$

Thus we reduced the solution of a nonlinear equation to solving several linear equations with the same boundary conditions. In physical terms this means that the problem of a beam on a one-sided support has been reduced to the problem of a beam on a two-sided support of variable stiffness the solution of which is already known.

Translated by L. K.